

アークタンジェント公式の代数的証明

2024年2月25日(日)

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1.はじめに

以前アークタンジェント公式を Proof Without Words 風に視覚的に証明しました。今回は文献[1]の補題を参考にして代数的に証明します。証明したい公式にひたすら同じ補題を適用していくだけです。

2.補題

a, b, x 及び y が正の実数で $a \leq b$ かつ $x < y$ を満たすとき,

$$\arctan \frac{a}{b} + \arctan \frac{x}{y} = \arctan \frac{ay + bx}{by - ax}$$

が成立する。

3. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{3} \Rightarrow (\text{右辺}) = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

4. $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$ の証明

$$\frac{a}{b} = \frac{1}{5}, \frac{x}{y} = \frac{1}{5} \Rightarrow 2 \arctan \frac{1}{5} = \arctan \frac{5}{12}$$

$$\frac{a}{b} = \frac{5}{12}, \frac{x}{y} = \frac{5}{12} \Rightarrow 4 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan \frac{120}{119}$$

一方で

$$\frac{a}{b} = \frac{1}{1}, \frac{x}{y} = \frac{1}{239} \Rightarrow \arctan \frac{1}{1} + \arctan \frac{1}{239} = \arctan \frac{120}{119}$$

よって

$$4 \arctan \frac{1}{5} = \arctan \frac{1}{1} + \arctan \frac{1}{239}$$

となり証明終了.

5. $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{3} \Rightarrow 2 \arctan \frac{1}{3} = \arctan \frac{3}{4}$$

$$\frac{a}{b} = \frac{3}{4}, \frac{x}{y} = \frac{1}{7} \Rightarrow (\text{右辺}) = \arctan \frac{3}{4} + \arctan \frac{1}{7} = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

6. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{5} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{5} = \arctan \frac{7}{9}$$

$$\frac{a}{b} = \frac{7}{9}, \frac{x}{y} = \frac{1}{8} \Rightarrow (\text{右辺}) = \arctan \frac{7}{9} + \arctan \frac{1}{8} = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

7. $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{7}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{2} \Rightarrow 2 \arctan \frac{1}{2} = \arctan \frac{4}{3}$$

$$\frac{a}{b} = \frac{1}{1}, \frac{x}{y} = \frac{1}{7} \Rightarrow \arctan \frac{1}{1} + \arctan \frac{1}{7} = \arctan \frac{4}{3}$$

以上より

$$2 \arctan \frac{1}{2} = \arctan \frac{1}{1} + \arctan \frac{1}{7}$$

となり証明終了.

8. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{4} + \arctan \frac{1}{13}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{4} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{4} = \arctan \frac{6}{7}$$

$$\frac{a}{b} = \frac{6}{7}, \frac{x}{y} = \frac{1}{13} \Rightarrow (\text{右辺}) = \arctan \frac{6}{7} + \arctan \frac{1}{13} = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

9. $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{12} + \arctan \frac{1}{17}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{3} \Rightarrow 2 \arctan \frac{1}{3} = \arctan \frac{3}{4}$$

$$\frac{a}{b} = \frac{3}{4}, \frac{x}{y} = \frac{1}{12} \Rightarrow \arctan \frac{3}{4} + \arctan \frac{1}{12} = \arctan \frac{8}{9}$$

$$\frac{a}{b} = \frac{8}{9}, \frac{x}{y} = \frac{1}{17} \Rightarrow (\text{右辺}) = \arctan \frac{8}{9} + \arctan \frac{1}{17} = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

10. $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{8} + \arctan \frac{1}{57}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{3} \Rightarrow 2 \arctan \frac{1}{3} = \arctan \frac{3}{4}$$

$$\frac{a}{b} = \frac{3}{4}, \frac{x}{y} = \frac{1}{8} \Rightarrow \arctan \frac{3}{4} + \arctan \frac{1}{8} = \arctan \frac{28}{29}$$

$$\frac{a}{b} = \frac{28}{29}, \frac{x}{y} = \frac{1}{57} \Rightarrow (\text{右辺}) = \arctan \frac{28}{29} + \arctan \frac{1}{57} = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

11. $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{9} + \arctan \frac{1}{32}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{3} \Rightarrow 2 \arctan \frac{1}{3} = \arctan \frac{3}{4}$$

$$\frac{a}{b} = \frac{3}{4}, \frac{x}{y} = \frac{1}{9} \Rightarrow \arctan \frac{3}{4} + \arctan \frac{1}{9} = \arctan \frac{31}{33}$$

$$\frac{a}{b} = \frac{31}{33}, \frac{x}{y} = \frac{1}{32} \Rightarrow (\text{右辺}) = \arctan \frac{31}{33} + \arctan \frac{1}{32} = \arctan \frac{1}{1} = (\text{左辺})$$

証明終了.

12. $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{5} + \arctan \frac{1}{18}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{18} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{18} = \arctan \frac{4}{7}$$

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{4}{7} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{4}{7} = \arctan \frac{3}{2}$$

以上より

$$2 \arctan \frac{1}{2} + \arctan \frac{1}{18} = \arctan \frac{3}{2}$$

一方で

$$\frac{a}{b} = \frac{1}{1}, \frac{x}{y} = \frac{1}{5} \Rightarrow \arctan \frac{1}{1} + \arctan \frac{1}{5} = \arctan \frac{3}{2}$$

よって

$$2 \arctan \frac{1}{2} + \arctan \frac{1}{18} = \arctan \frac{1}{1} + \arctan \frac{1}{5}$$

となり証明終了.

13. $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{9} - \arctan \frac{1}{32}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{2} \Rightarrow 2 \arctan \frac{1}{2} = \arctan \frac{4}{3}$$

一方で

$$\frac{a}{b} = \frac{1}{9}, \frac{x}{y} = \frac{1}{32} \Rightarrow \arctan \frac{1}{9} + \arctan \frac{1}{32} = \arctan \frac{1}{7}$$

$$\frac{a}{b} = \frac{1}{1}, \frac{x}{y} = \frac{1}{7} \Rightarrow \arctan \frac{1}{1} + \arctan \frac{1}{7} = \arctan \frac{4}{3}$$

よって

$$\arctan \frac{1}{1} + \arctan \frac{1}{9} + \arctan \frac{1}{32} = \arctan \frac{4}{3}$$

以上より

$$2 \arctan \frac{1}{2} = \arctan \frac{1}{1} + \arctan \frac{1}{9} + \arctan \frac{1}{32}$$

となり証明終了.

14. $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{6} + \arctan \frac{1}{43}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{43} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{43} = \arctan \frac{9}{17}$$

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{9}{17} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{9}{17} = \arctan \frac{7}{5}$$

以上より

$$2 \arctan \frac{1}{2} + \arctan \frac{1}{43} = \arctan \frac{7}{5}$$

一方で

$$\frac{a}{b} = \frac{1}{1}, \frac{x}{y} = \frac{1}{6} \Rightarrow \arctan \frac{1}{1} + \arctan \frac{1}{6} = \arctan \frac{7}{5}$$

以上より

$$2 \arctan \frac{1}{2} + \arctan \frac{1}{43} = \arctan \frac{1}{1} + \arctan \frac{1}{6}$$

となり証明終了.

15. $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{8} - \arctan \frac{1}{57}$ の証明

$$\frac{a}{b} = \frac{x}{y} = \frac{1}{2} \Rightarrow 2 \arctan \frac{1}{2} = \arctan \frac{4}{3}$$

一方で

$$\frac{a}{b} = \frac{1}{8}, \frac{x}{y} = \frac{1}{57} \Rightarrow \arctan \frac{1}{8} + \arctan \frac{1}{57} = \arctan \frac{1}{7}$$

$$\frac{a}{b} = \frac{1}{1}, \frac{x}{y} = \frac{1}{7} \Rightarrow \arctan \frac{1}{1} + \arctan \frac{1}{7} = \arctan \frac{4}{3}$$

よって

$$\arctan \frac{1}{1} + \arctan \frac{1}{8} + \arctan \frac{1}{57} = \arctan \frac{4}{3}$$

したがって

$$2 \arctan \frac{1}{2} = \arctan \frac{1}{1} + \arctan \frac{1}{8} + \arctan \frac{1}{57}$$

となり証明終了.

16. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{13} + \arctan \frac{1}{21}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{5} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{5} = \arctan \frac{7}{9}$$

$$\frac{a}{b} = \frac{7}{9}, \frac{x}{y} = \frac{1}{13} \Rightarrow \arctan \frac{7}{9} + \arctan \frac{1}{13} = \arctan \frac{10}{11}$$

$$\frac{a}{b} = \frac{10}{11}, \frac{x}{y} = \frac{1}{21} \Rightarrow (\text{右辺}) = \arctan \frac{10}{11} + \arctan \frac{1}{21} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了.

17. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{4} + \arctan \frac{1}{23} + \arctan \frac{1}{30}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{4} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{4} = \arctan \frac{6}{7}$$

$$\frac{a}{b} = \frac{6}{7}, \frac{x}{y} = \frac{1}{23} \Rightarrow \arctan \frac{6}{7} + \arctan \frac{1}{23} = \arctan \frac{29}{31}$$

$$\frac{a}{b} = \frac{29}{31}, \frac{x}{y} = \frac{1}{30} \Rightarrow (\text{右辺}) = \arctan \frac{29}{31} + \arctan \frac{1}{30} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了.

18. $\frac{\pi}{4} = \arctan \frac{1}{3} + \arctan \frac{1}{5} + \arctan \frac{1}{7} + \arctan \frac{1}{8}$ の証明

$$\frac{a}{b} = \frac{1}{3}, \frac{x}{y} = \frac{1}{5} \Rightarrow \arctan \frac{1}{3} + \arctan \frac{1}{5} = \arctan \frac{4}{7}$$

$$\frac{a}{b} = \frac{4}{7}, \frac{x}{y} = \frac{1}{7} \Rightarrow \arctan \frac{4}{7} + \arctan \frac{1}{7} = \arctan \frac{7}{9}$$

$$\frac{a}{b} = \frac{7}{9}, \frac{x}{y} = \frac{1}{8} \Rightarrow (\text{右辺}) = \arctan \frac{7}{9} + \arctan \frac{1}{8} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了.

19. $\frac{\pi}{4} = \arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{7} + \arctan \frac{1}{13}$ の証明

$$\frac{a}{b} = \frac{1}{3}, \frac{x}{y} = \frac{1}{4} \Rightarrow \arctan \frac{1}{3} + \arctan \frac{1}{4} = \arctan \frac{7}{11}$$

$$\frac{a}{b} = \frac{7}{11}, \frac{x}{y} = \frac{1}{7} \Rightarrow \arctan \frac{7}{11} + \arctan \frac{1}{7} = \arctan \frac{6}{7}$$

$$\frac{a}{b} = \frac{6}{7}, \frac{x}{y} = \frac{1}{13} \Rightarrow (\text{右辺}) = \arctan \frac{6}{7} + \arctan \frac{1}{13} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了.

20. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{9} + \arctan \frac{1}{73}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{5} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{5} = \arctan \frac{7}{9}$$

$$\frac{a}{b} = \frac{7}{9}, \frac{x}{y} = \frac{1}{9} \Rightarrow \arctan \frac{7}{9} + \arctan \frac{1}{9} = \arctan \frac{36}{37}$$

$$\frac{a}{b} = \frac{36}{37}, \frac{x}{y} = \frac{1}{73} \Rightarrow (\text{右辺}) = \arctan \frac{36}{37} + \arctan \frac{1}{73} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了.

21. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{4} + \arctan \frac{1}{15} + \arctan \frac{1}{98}$ の証明

$$\frac{a}{b} = \frac{1}{2}, \frac{x}{y} = \frac{1}{4} \Rightarrow \arctan \frac{1}{2} + \arctan \frac{1}{4} = \arctan \frac{6}{7}$$

$$\frac{a}{b} = \frac{6}{7}, \frac{x}{y} = \frac{1}{15} \Rightarrow \arctan \frac{6}{7} + \arctan \frac{1}{15} = \arctan \frac{97}{99}$$

$$\frac{a}{b} = \frac{97}{99}, \frac{x}{y} = \frac{1}{98} \Rightarrow (\text{右辺}) = \arctan \frac{97}{99} + \arctan \frac{1}{98} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了。

22. $\frac{\pi}{4} = \arctan \frac{1}{7} + \arctan \frac{1}{8} + \arctan \frac{4}{7}$ の証明

$$\frac{a}{b} = \frac{1}{7}, \frac{x}{y} = \frac{1}{8} \Rightarrow \arctan \frac{1}{7} + \arctan \frac{1}{8} = \arctan \frac{3}{11}$$

$$\frac{a}{b} = \frac{3}{11}, \frac{x}{y} = \frac{4}{7} \Rightarrow (\text{右辺}) = \arctan \frac{3}{11} + \arctan \frac{4}{7} = \arctan \frac{1}{1} = (\text{左辺})$$

となり証明終了。

参考文献

- [1] R. Nelsen, “Machin’s Formula Via a Proof Without Words”, The College Mathematics Journal, vol. 52, no. 5, p. 355, 2021.